eq^n (4) & (6) become

$$-\frac{dV_s}{dz} = (R+j\omega L)\, I_s \quad \text{—— (8)}$$

$$-\frac{dI_s}{dz} = (G+j\omega C)\, V_s \quad \text{—— (9)}$$

Eq^n (8) & (9) are coupled

Taking second derivative to separate them

$$\frac{d^2V_s}{dz^2} = (R+j\omega L)(G+j\omega C)\, V_s$$

$$\Rightarrow \quad \frac{d^2V_s}{dz^2} - \gamma^2 V_s = 0 \quad \text{—— (10)}$$

where $\boxed{\gamma = \alpha + j\beta = \sqrt{(R+j\omega L)(G+j\omega C)}}$ —— (11)

taking the second derivative of I_s in eq^n (9) and employing eq^n (8)

$$\frac{d^2 I_s}{d\cancel{z}^2} - \gamma^2 I_s = 0 \quad \text{—— (12)}$$

Eq^n (10) & (12) are → wave eq^ns for voltage and current → similar to the wave eq^ns obtained for plane waves

γ in eq^n (11) → propagation constant (in per meter), α → attenuation constant (in nepers per meter or decibel per meter)

$\boxed{= 8.68 \text{ dB}}$

β → phase constant (in radians per meter)

The wavelength λ & wave velocity v are given by

$$\lambda = \frac{2\pi}{\beta} \quad -(13), \qquad \boxed{v = \frac{\omega}{\beta} = f\lambda} \quad -(14)$$

Solⁿs of homogeneous differential eqⁿ (10) & (12)

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad -(15)$$

$$\xrightarrow{+z} \qquad \xleftarrow{-z}$$

and

$$I_s(z) = I_0^+ e^{-\gamma z} + I_0^- e^{\gamma z} \quad -(16)$$

$$\xrightarrow{+z} \qquad \xleftarrow{-z}$$

V_0^+, V_0^-, I_0^+, I_0^- → wave amplitude

+, − → wave traveling along +z and −z directions.

⇒ Instantaneous expression for voltage as

$$V(z,t) = Re\left[V_s(z)\, e^{j\omega t}\right]$$

$$= V_0^+ e^{-\alpha z} \cos(\omega t - \beta z) + V_0^- e^{\alpha z} \cos(\omega t + \beta z) \quad -(17)$$

Characteristic impedance Z_0 of the line

↳ ratio of positively traveling voltage wave to the current wave at any point on the line.

Z_0 → analogous to the η → intrinsic impedance of the medium of wave propagation.

Substituting eq^n (15) and (16) in eq^n (8) & (9) and equating coefficient of $e^{\gamma z}$ and $e^{-\gamma z}$

$$Z_0 = \frac{V_0^+}{I_0^+} = -\frac{V_0^-}{I_0^-} = \frac{R+j\omega L}{\gamma} = \frac{\gamma}{G+j\omega C} \quad \text{--- (18)}$$

or

$$Z_0 = \sqrt{\frac{R+j\omega L}{G+j\omega C}} = \underbrace{R_0}_{\text{Real}} + j\underbrace{X_0}_{\text{imaginary part of } Z_0} \quad \text{--- (19)}$$

$\underbrace{R_0}_{\text{ohms}}$ not to be mistaken for $\underbrace{R}_{\text{ohms/meter}}$

$\left.\begin{matrix}\gamma \\ Z_0\end{matrix}\right\} \longrightarrow$ important properties of the line

$\longrightarrow$ Both depend on the line parameters R, L, G and C and frequency of operation.

admittance $Y_0 = \dfrac{1}{Z_0}$.

We have considered lossy type transmission line $\longrightarrow$ The conductors comprising the line are imperfect ($\sigma_c \neq \infty$) and the dielectric in which the conductors are embedded is lossy. ($\sigma \neq 0$)

Lossless Line ($R = 0 = G$)

A transmission line → lossless

if the conductors of the line are perfect ($\sigma_c \simeq \infty$) and the dielectric medium separating them is lossless ($\sigma = 0$)

when $\sigma_c \simeq \infty$, $\sigma \simeq 0$

$$\boxed{R = 0 = G} \quad \text{--- (20)}$$

↓

Necessary conditions for a line to be lossless.

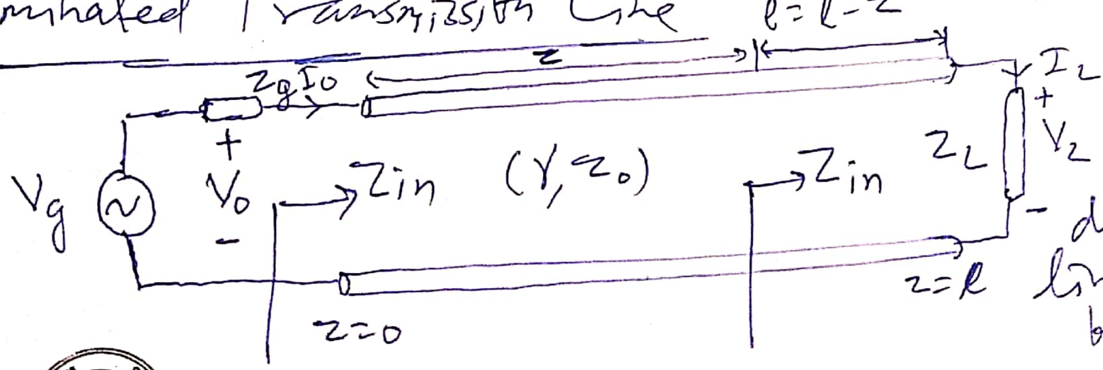
For such a line, eq^n (20) forces eq^n (11), (14) and (15) to become

$$\alpha = 0, \quad \gamma = j\beta = j\omega\sqrt{LC} \quad \text{--- 21 (a)}$$

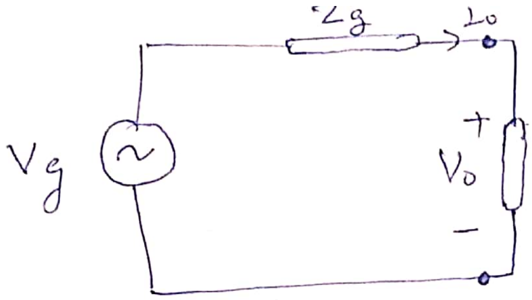
$$v = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}} = f\lambda \quad \text{--- 21 (b)}$$

$$X_0 = 0, \quad Z_0 = R_0 = \sqrt{\frac{L}{C}} \quad \text{--- 21 (c)}$$

Terminated Transmission Line



Input impedance due to a line terminated by a load



Equivalent circuit for finding V_0 and I_0 in terms of Z_{in} at input

Transmission line of length l characterised by γ and Z_0, connected to a load Z_L. The generator sees the line with the load as an input impedance Z_{in}

Transmission line extend from $z=0$ at the generator to $z=l$ at the load.

The voltage and current eq^n in (15) & (16) incorporating eq^n (18)

$$V_s(z) = V_0^+ e^{-\gamma z} + V_0^- e^{\gamma z} \quad \text{--- (22)}$$

$$I_s(z) = \frac{V_0^+}{Z_0} e^{-\gamma z} - \frac{V_0^-}{Z_0} e^{\gamma z} \quad \text{--- (23)}$$

$\left.\begin{matrix} V_0^+ \\ V_0^- \end{matrix}\right\}$? Terminal conditions must be given

if we are given the conditions at the input

say $\quad V_0 = V(z=0), \quad I_0 = I(z=0)$ --- (24)

Substituting it into eq^n (22), (23)

$$V_0^+ = \frac{1}{2}(V_0 + Z_0 I_0) \quad \text{--- 25 (a)}$$

$$V_0^- = \frac{1}{2}(V_0 - Z_0 I_0) \quad \text{--- 25 (b)}$$

If the input impedance at the input terminal is Z_{in}, the input voltage V_0 and the input current I_0 are obtained as

$$V_0 = \frac{Z_{in}}{Z_{in}+Z_g} V_g \quad , \quad I_0 = \frac{V_g}{Z_{in}+Z_g} \quad \text{--- (26)}$$

If we are given the conditions at the load, say

$$V_L = V(z=l) \, , \quad I_L = I(z=l) \quad \text{--- (27)}$$

from eq^n (22) & (23)

$$V_0^+ = \frac{1}{2}(V_L + Z_0 I_L) e^{\gamma l} \quad \text{--- 28 (a)}$$

$$V_0^- = \frac{1}{2}(V_L - Z_0 I_L) e^{-\gamma l} \quad \text{--- 28 (b)}$$

We find the input impedance $Z_{in} = V_s(z)/I_s(z)$ at any point on the line. At the generator, for example eq^n (22) and (23) yield

$$Z_{in} = \frac{V_s(z)}{I_s(z)} = \frac{Z_0 (V_0^+ + V_0^-)}{V_0^+ + V_0^-} \quad \text{--- (29)}$$

from eq^n (28) & (29) and

$$\frac{e^{\gamma l} + e^{-\gamma l}}{2} = \cosh \gamma l \; ; \quad \frac{e^{\gamma l} - e^{-\gamma l}}{2} = \sinh \gamma l \quad \text{--- 30 (a)}$$

or

$$\tanh \gamma l = \frac{\sinh \gamma l}{\cosh \gamma l} = \frac{e^{\gamma l} - e^{-\gamma l}}{e^{\gamma l} + e^{-\gamma l}} \quad \text{--- 30 (b)}$$

$$\Rightarrow \quad Z_{in} = Z_0 \left[\frac{Z_L + Z_0 \tanh \gamma l}{Z_0 + Z_L \tanh \gamma l} \right] (\text{Lossy}) \quad \text{--- (31)}$$